

INEQUALITIES OF DIRICHLET EIGENVALUES FOR DEGENERATE ELLIPTIC PARTIAL DIFFERENTIAL OPERATORS

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ABSTRACT. Let $X_j, Y_j (j = 1, \dots, n)$ be vector fields satisfying Hörmander's condition and $\Delta_L = \sum_{j=1}^n (X_j^2 + Y_j^2)$. In this paper, we establish some inequalities of Dirichlet eigenvalues for degenerate elliptic partial differential operator Δ_L and Δ_L^2 . These inequalities extend Yang's inequalities for Dirichlet eigenvalues of Laplacian to the settings here and the forms of inequalities are more general than Yang's inequalities. To obtain them, we give a generalization of the inequality by Chebyshev.

1. INTRODUCTION

Estimates of Dirichlet eigenvalues for Laplacian in the Euclidean space have been extensively studied. For the following Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in R^n , Payne, Pólya and Weinberger in [11] obtained the inequality (now called the PPW inequality)

$$\lambda_{k+1} - \lambda_k \leq \frac{4}{nk} \sum_{r=1}^k \lambda_r.$$

Hile and Protter in [4] proved the inequality (now called the HP inequality)

$$\sum_{r=1}^k \frac{\lambda_r}{\lambda_{k+1} - \lambda_r} \geq \frac{nk}{4}.$$

Recently, Yang in [13] established some important eigenvalue estimates including Yang's first inequality

$$\sum_{r=1}^k (\lambda_{k+1} - \lambda_r)^2 \leq \frac{4}{n} \sum_{r=1}^k (\lambda_{k+1} - \lambda_r) \lambda_r$$

1991 *Mathematics Subject Classification.* Primary 35H05, Secondary 35P15.

Key words and phrases. degenerate elliptic partial differential operator; Dirichlet eigenvalue; inequality.

This work was supported by the National Natural Science Foundation of China (Grant Nos. 11271299, 11001221) and Natural Science Foundation Research Project of Shaanxi Province (2012JM1014).

and Yang's second inequality

$$\lambda_{k+1} \leq \left(1 + \frac{4}{n}\right) \frac{1}{k} \sum_{r=1}^k \lambda_r.$$

Some estimates for Dirichlet eigenvalues of sub-Laplacian on the Heisenberg group was deduced. Niu and Zhang in [10] obtained the PPW type inequality:

$$\lambda_{k+1} - \lambda_k \leq \frac{2}{nk} \left(\sum_{r=1}^k \lambda_r \right).$$

Ilias and Makhoul in [5] gave the Yang type inequalities.

In the paper, we consider the following two Dirichlet problems:

$$(1.1) \quad \begin{cases} -\Delta_L u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

and

$$(1.2) \quad \begin{cases} (-\Delta_L)^2 u = \lambda u, & \text{in } \Omega, \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset R^{2n+1}$ is a bounded domain, the boundary $\partial\Omega$ is smooth and not characteristic, ν is the outward unit normal on $\partial\Omega$; Δ_L is the degenerate elliptic partial differential operator constituted by vector fields $X_j, Y_j (j = 1, \dots, n)$ satisfying Hörmander's condition,

$$(1.3) \quad \Delta_L = \sum_{j=1}^n (X_j^2 + Y_j^2),$$

where $X_j = \frac{\partial}{\partial x_j} + 2\sigma y_j |z|^{2\sigma-2} \frac{\partial}{\partial t}$, $Y_j = \frac{\partial}{\partial y_j} - 2\sigma x_j |z|^{2\sigma-2} \frac{\partial}{\partial t}$, $j = 1, \dots, n$, $x, y \in R^n$,

$t \in R$, $z = x + \sqrt{-1}y \in C$, $|z| = \left[\sum_{j=1}^n (x_j^2 + y_j^2) \right]^{\frac{1}{2}}$, σ is any natural number. When

$\sigma = 1$, Δ_L is the sub-Laplacian on the Heisenberg group; when $\sigma = 2, 3, \dots$, Δ_L is the operators discussed by Greiner (see [3, 8]). We note that compared with sub-Laplacian on the Heisenberg group, those operators by Greiner do not have properties of group structure and translation. Some related papers see [9, 14].

From [7], we know that the eigenvalues of (1.1) and (1.2) exist and satisfy

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots \rightarrow +\infty.$$

The corresponding orthogonal normalized eigenfunctions $u_1, u_2, \dots, u_k, \dots$ satisfy $\langle u_i, u_l \rangle = \delta_{il}$, $i, l = 1, 2, \dots$. Since the boundary $\partial\Omega$ is not characteristic, the eigenfunctions are smooth by using the results in [12].

For convenience, we denote $L = -\Delta_L$ in the sequel. The main results of this paper are the following:

Theorem 1.1. *Let $\{\lambda_i\}$ be the eigenvalues of (1.1), then*

$$(1.4) \quad \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \leq \sqrt{\frac{2}{n}} \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\beta \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \lambda_i \right)^{\frac{1}{2}}.$$

where $\alpha \in R, \beta \geq 0$ and $\alpha^2 \leq 2\beta$.

Inequality (1.4) is the generalization of Yang Type inequalities. Using Theorem 1, it follows some interesting corollaries.

Corollary 1.2. *Let $\{\lambda_i\}$ be the eigenvalues of (1.1), then we have the Yang type first inequality*

$$(1.5) \quad \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{2}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i.$$

Corollary 1.3. *Let $\{\lambda_i\}$ be the eigenvalues of (1.1), then we have the Payne-Pólya-Weinberger Type inequality*

$$(1.6) \quad \lambda_{k+1} - \lambda_k \leq \frac{2}{nk} \sum_{i=1}^k \lambda_i.$$

Corollary 1.4. *Let $\{\lambda_i\}$ be the eigenvalues of (1.1), then we have the Yang type second inequality*

$$(1.7) \quad \lambda_{k+1} \leq \left(1 + \frac{2}{n}\right) \frac{1}{k} \left(\sum_{i=1}^k \lambda_i\right).$$

Theorem 1.5. *Let $\{\lambda_i\}$ be the eigenvalues of (1.2), then*

$$(1.8) \quad \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \leq \frac{2\sqrt{n+1}}{n} \left[\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\beta \lambda_i^{\frac{1}{2}} \right]^{\frac{1}{2}} \left[\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \lambda_i^{\frac{1}{2}} \right]^{\frac{1}{2}}.$$

where $\alpha \in R, \beta \geq 0$, and $\alpha^2 \leq 2\beta$.

Corollary 1.6. *Let $\{\lambda_i\}$ be the eigenvalues of (1.2), then*

$$(1.9) \quad \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{2\sqrt{n+1}}{n} \left[\sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i^{\frac{1}{2}} \right]^{\frac{1}{2}} \left[\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \lambda_i^{\frac{1}{2}} \right]^{\frac{1}{2}}.$$

Corollary 1.7. *Let $\{\lambda_i\}$ be the eigenvalues of (1.2), then*

$$(1.10) \quad \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \leq \frac{2\sqrt{n+1}}{n} \left[\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\beta \right]^{\frac{1}{2}} \left[\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \lambda_i \right]^{\frac{1}{2}}.$$

where $\alpha \in R, \beta \geq 0$, and $\alpha^2 \leq 2\beta$.

Corollary 1.8. *Let $\{\lambda_i\}$ be the eigenvalues of (1.2), then we have*

$$(1.11) \quad \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^2 \leq \frac{4(n+1)}{n^2} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \lambda_i.$$

Corollary 1.9. *Let $\{\lambda_i\}$ be the eigenvalues of (1.2), then we have*

$$(1.12) \quad \lambda_{k+1} - \lambda_k \leq \frac{4(n+1)}{n^2 k^2} \left(\sum_{i=1}^k \lambda_i^{\frac{1}{2}} \right)^2.$$

These results are new even for Laplacian on the Euclidean space and sub-Laplacian on the Heisenberg group.

This paper is arranged as follows. In Section 2 the definition of function couple χ_λ and its properties are given; two elementary inequalities (see Lemmas 2.6 and 2.8) are proved and examples of noncharacteristics and characteristics domains for vector fields are listed. The proofs of Theorem 1.1 and Corollaries 1.2-1.4 are put in Section 3. The proofs of Theorem 1.5 and Corollaries 1.6-1.9 are given in Section 4.

2. PRELIMINARY RESULTS

Definition 2.1. (see [5]) A couple (f, g) of functions on the interval $(0, \lambda)$ ($\lambda > 0$) is said to belong to χ_λ provided that

- (i) f and g are positive.
- (ii) f and g satisfy

$$\left(\frac{f(x) - f(y)}{x - y} \right)^2 + \left(\frac{(f(x))^2}{g(x)(\lambda - x)} + \frac{(f(y))^2}{g(y)(\lambda - y)} \right) \left(\frac{g(x) - g(y)}{x - y} \right) \leq 0,$$

for any $x, y \in (0, \lambda)$, $x \neq y$.

Lemma 2.2. *Let $(f, g) \in \chi_\lambda$, then g must be nonincreasing; if $f(x) = (\lambda - x)^\alpha$, $g(x) = (\lambda - x)^\beta$, then $\alpha^2 \leq 2\beta$.*

Proof. From Definition 2.1 we see that g must be nonincreasing. Because f and g satisfy

$$\left(\frac{f(x) - f(y)}{x - y} \right)^2 + \left(\frac{(f(x))^2}{g(x)(\lambda - x)} + \frac{(f(y))^2}{g(y)(\lambda - y)} \right) \left(\frac{g(x) - g(y)}{x - y} \right) \leq 0,$$

letting $y \rightarrow x$, we have

$$(f'(x))^2 + \frac{2(f(x))^2}{g(x)(\lambda - x)} g'(x) \leq 0$$

and then

$$\left(\frac{f'(x)}{f(x)} \right)^2 + \frac{2}{(\lambda - x)} \frac{g'(x)}{g(x)} \leq 0.$$

Taking $f(x) = (\lambda - x)^\alpha$, $g(x) = (\lambda - x)^\beta$, it follows $\alpha^2 \leq 2\beta$. \square

Definition 2.3. (see [5]) For any two operators A and B , their commutator $[A, B]$ is defined by $[A, B] = AB - BA$.

Lemma 2.4. *For $p = 1, 2, \dots, n$, we have*

$$(2.1) \quad L(x_p u_i) = x_p L u_i - 2X_p u_i,$$

$$(2.2) \quad [L, x_p] u_i = -2X_p u_i.$$

Proof. A direct calculation gives

$$\begin{aligned} X_j(x_p u_i) &= (X_j x_p) u_i + x_p (X_j u_i), \\ X_j^2(x_p u_i) &= X_j((X_j x_p) u_i + x_p (X_j u_i)) \\ &= 2(X_j x_p)(X_j u_i) + x_p (X_j^2 u_i). \end{aligned}$$

and

$$\begin{aligned} Y_j(x_p u_i) &= x_p (Y_j u_i), \\ Y_j^2(x_p u_i) &= x_p (Y_j^2 u_i). \end{aligned}$$

So

$$\begin{aligned} L(x_p u_i) &= - \sum_{j=1}^n (X_j^2 + Y_j^2)(x_p u_i) \\ &= - \sum_{j=1}^n [2(X_j x_p)(X_j u_i) + x_p (X_j^2 u_i) + x_p (Y_j^2 u_i)] \\ &= x_p L u_i - 2X_p u_i, \end{aligned}$$

and (2.1) is proved. Noting

$$[L, x_p] u_i = L(x_p u_i) - x_p L u_i = x_p L u_i - 2X_p u_i - x_p L u_i = -2X_p u_i,$$

(2.2) is proved. \square

Lemma 2.5. (see [5]) Let $A: D \subset H \rightarrow H$ be a self-adjoint operator defined on a dense domain D , which is semibounded below and has a discrete spectrum $\lambda_1 \leq \lambda_2 \leq \lambda_3 \dots$. Let $\{T_p: D \rightarrow H\}_{p=1}^n$ be a collection of skew-symmetric operators, and $\{B_p: T_p(D) \rightarrow H\}_{p=1}^n$ be a collection of symmetric operators, leaving D invariant. We denote by $\{u_i\}_{i=1}^n$ a basis of orthonormal eigenvectors of A , u_i corresponding to λ_i . Let $k \geq 1$ and assume $\lambda_{k+1} \geq \lambda_k$. Then for any (f, g) in $\chi_{\lambda_{k+1}}$, it follows

$$\begin{aligned} (2.3) \quad & \left(\sum_{i=1}^k \sum_{p=1}^n f(\lambda_i) \langle [T_p, B_p] u_i, u_i \rangle \right)^2 \\ & \leq 4 \left(\sum_{i=1}^k \sum_{p=1}^n g(\lambda_i) \langle [A, B_p] u_i, B_p u_i \rangle \right) \left(\sum_{i=1}^k \sum_{p=1}^n \frac{(f(\lambda_i))^2}{g(\lambda_i)(\lambda_{k+1} - \lambda_i)} \|T_p u_i\|^2 \right). \end{aligned}$$

Lemma 2.6. For $\gamma \geq 1$, $s_i \geq 0, i = 1, \dots, k$, we have

$$\left(\sum_{i=1}^k s_i \right)^\gamma \leq k^{\gamma-1} \sum_{i=1}^k s_i^\gamma.$$

Proof. Let $\theta(s) = s^\gamma, s \geq 0, \gamma \geq 1$, so $\theta'(s) = \gamma s^{\gamma-1} \geq 0, \theta''(s) = \gamma(\gamma-1)s^{\gamma-2} \geq 0$. Noting that $\theta(s)$ is a convex function on $(0, +\infty)$, we have that for $s_i > 0, i = 1, \dots, k$, it holds

$$\theta \left(\frac{\sum_{i=1}^k s_i}{k} \right) \leq \frac{\sum_{i=1}^k \theta(s_i)}{k},$$

and yields

$$\left(\frac{\sum_{i=1}^k s_i}{k} \right)^\gamma \leq \frac{\sum_{i=1}^k s_i^\gamma}{k}.$$

The required inequality is proved. \square

Lemma 2.7. (*Chebyshev's inequality, [6]*) If $(a_k - a_j)(b_k - b_j) \leq 0$ for any non-negative k, j , then

$$\sum_{i=1}^n a_i b_i \leq \frac{1}{n} \left(\sum_{i=1}^n a_i \right) \left(\sum_{i=1}^n b_i \right).$$

A key preliminary inequality in the paper is the following which enable us to obtain estimates of eigenvalues more general than Yang's.

Lemma 2.8. If $A_1 \geq A_2 \geq \cdots \geq A_k \geq 0$, $0 \leq B_1 \leq B_2 \leq \cdots \leq B_k$, $0 \leq C_1 \leq C_2 \leq \cdots \leq C_k$, $i = 1, \dots, k$, then for $\alpha^2 \leq 2\beta$, we have

$$(2.4) \quad \sum_{i=1}^k A_i^\beta B_i \sum_{i=1}^k A_i^{2\alpha-\beta-1} C_i \leq \sum_{i=1}^k A_i^\beta \sum_{i=1}^k A_i^{2\alpha-\beta-1} B_i C_i.$$

Proof. When $k = 1$, we see that (2.4) is true, since $A_1^\beta B_1 A_1^{2\alpha-\beta-1} C_1 - A_1^\beta A_1^{2\alpha-\beta-1} B_1 C_1 = 0$. Now suppose that the conclusion is true for $k - 1$, then

$$\begin{aligned} & \sum_{i=1}^k A_i^\beta B_i \sum_{i=1}^k A_i^{2\alpha-\beta-1} C_i - \sum_{i=1}^k A_i^\beta \sum_{i=1}^k A_i^{2\alpha-\beta-1} B_i C_i \\ &= \sum_{i=1}^{k-1} A_i^\beta B_i \sum_{i=1}^{k-1} A_i^{2\alpha-\beta-1} C_i - \sum_{i=1}^{k-1} A_i^\beta \sum_{i=1}^{k-1} A_i^{2\alpha-\beta-1} B_i C_i \\ & \quad + A_k^{2\alpha-1} B_k C_k - A_k^{2\alpha-1} B_k C_k \\ & \quad + A_k^{2\alpha-\beta-1} C_k \sum_{i=1}^{k-1} A_i^\beta B_i + A_k^\beta B_k \sum_{i=1}^{k-1} A_i^{2\alpha-\beta-1} C_i \\ & \quad - A_k^{2\alpha-\beta-1} B_k C_k \sum_{i=1}^{k-1} A_i^\beta - A_k^\beta \sum_{i=1}^{k-1} A_i^{2\alpha-\beta-1} B_i C_i. \end{aligned}$$

Based on the assumption for $k - 1$, we have

$$\begin{aligned} (2.5) \quad & \sum_{i=1}^k A_i^\beta B_i \sum_{i=1}^k A_i^{2\alpha-\beta-1} C_i - \sum_{i=1}^k A_i^\beta \sum_{i=1}^k A_i^{2\alpha-\beta-1} B_i C_i \\ & \leq A_k^{2\alpha-\beta-1} C_k \sum_{i=1}^{k-1} A_i^\beta (B_i - B_k) - A_k^\beta \sum_{i=1}^{k-1} A_i^{2\alpha-\beta-1} C_i (B_i - B_k) \\ & = A_k^{2\alpha-\beta-1} \sum_{i=1}^{k-1} A_i^{2\alpha-\beta-1} \left(A_i^{2\beta-2\alpha+1} C_k - A_k^{2\beta-2\alpha+1} C_i \right) (B_i - B_k). \end{aligned}$$

Noting $\alpha^2 \leq 2\beta$ and $2\alpha - 1 \leq \alpha^2$ implies $2\beta - 2\alpha + 1 \geq 0$.

If $A_1 \geq A_2 \geq \cdots \geq A_k > 0$, $0 < B_1 \leq B_2 \leq \cdots \leq B_k$, $0 < C_1 \leq C_2 \leq \cdots \leq C_k$, then for $i = 1, \dots, k$,

$$\frac{A_i^{2\beta-2\alpha+1} C_k}{A_k^{2\beta-2\alpha+1} C_i} = \left(\frac{C_k}{C_i} \right) \left(\frac{A_i}{A_k} \right)^{2\beta-2\alpha+1} \geq 1, \frac{B_i}{B_k} \leq 1.$$

and

$$(2.6) \quad A_i^{2\beta-2\alpha+1}C_k - A_k^{2\beta-2\alpha+1}C_i \geq 0, B_i - B_k \leq 0.$$

If A_i, B_i and $C_i, i = 1, \dots, k$, are nonnegative, then (2.6) is also true. Hence from (2.6),

$$\sum_{i=1}^k A_i^\beta B_i \sum_{i=1}^k A_i^{2\alpha-\beta-1} C_i - \sum_{i=1}^k A_i^\beta \sum_{i=1}^k A_i^{2\alpha-\beta-1} B_i C_i \leq 0,$$

and (2.4) is proved. \square

By Lemma 2.8, we immediately have the following result proved in [1].

Corollary 2.9. *If $A_1 \geq A_2 \geq \dots \geq A_k \geq 0$, $0 \leq B_1 \leq B_2 \leq \dots \leq B_k$, $0 \leq C_1 \leq C_2 \leq \dots \leq C_k$, $i = 1, \dots, k$, then we have*

$$\sum_{i=1}^n A_i^2 B_i \sum_{i=1}^n A_i C_i \leq \sum_{i=1}^n A_i^2 \sum_{i=1}^n A_i B_i C_i.$$

Now let us describe some characteristic and noncharacteristic domains with respect to vector fields and give some such domains.

Definition 2.10. Let $\phi(z, t)$ be the boundary function of a domain Ω . We call that a point (z, t) on $\partial\Omega$ is a characteristic point with respect to vector fields X_j, Y_j ($j = 1, \dots, n$), if it satisfies $|\nabla_L \phi(z, t)| = 0$, where $\nabla_L = (X_1, \dots, X_n, Y_1, \dots, Y_n)$. A domain with characteristic points is called a characteristic domain. If the boundary $\partial\Omega$ does not have any characteristic point, then Ω is said a noncharacteristic domain.

Proposition 2.11. *The sets $\Omega_m = \{(z, t) \in C^{2n} \times R \mid (|z| - a)^2 + (t - b)^2 < m^2\}$, $m = 1, 2, \dots$, are noncharacteristic domains with respect to X_j, Y_j ($j = 1, \dots, n$), where $a > 0$, b is any real number.*

Proof. Fix m and denote $\psi(z, t) = (|z| - a)^2 + (t - b)^2 - m^2$, then

$$\begin{aligned} X_j \psi(z, t) &= \frac{\partial}{\partial x_j} \left((|z| - a)^2 + (t - b)^2 - m^2 \right) + 2\sigma y_j |z|^{2\sigma-2} \frac{\partial}{\partial t} \left((|z| - a)^2 + (t - b)^2 - m^2 \right) \\ &= \frac{2(|z| - a)x_j}{|z|} + 4\sigma y_j |z|^{2\sigma-2} (t - b), \end{aligned}$$

$$\begin{aligned} Y_j \psi(z, t) &= \frac{\partial}{\partial y_j} \left((|z| - a)^2 + (t - b)^2 - m^2 \right) - 2\sigma x_j |z|^{2\sigma-2} \frac{\partial}{\partial t} \left((|z| - a)^2 + (t - b)^2 - m^2 \right) \\ &= \frac{2(|z| - a)y_j}{|z|} - 4\sigma x_j |z|^{2\sigma-2} (t - b) \end{aligned}$$

and

$$\begin{aligned} |\nabla_L \psi(z, t)|^2 &= \sum_{j=1}^n \left(|X_j \psi(z, t)|^2 + |Y_j \psi(z, t)|^2 \right) \\ &= \sum_{j=1}^n \left(\frac{4(|z| - a)^2 (x_j^2 + y_j^2)}{|z|^2} + 16\sigma^2 |z|^{4\sigma-4} (t - b)^2 (x_j^2 + y_j^2) \right) \\ &= 4(|z| - a)^2 + 16\sigma^2 |z|^{4\sigma-2} (t - b)^2. \end{aligned}$$

If $|\nabla_L \psi(z, t)| = 0$, then $|z| = a, t = b$. But points satisfying these conditions do not be on the boundary $\partial\Omega_m, m = 1, 2, \dots$, so $\Omega_m, m = 1, 2, \dots$, are noncharacteristic. \square

If we take $a = 2, b = 0$, then (see [2] for the case of Heisenberg groups)

Corollary 2.12. *The sets $\Omega_m = \{(z, t) \in C^{2n} \times R \mid (|z| - 2)^2 + t^2 < m^2\}$, $m = 1, 2, \dots$, are noncharacteristic domains with respect to vector fields $X_j, Y_j (j = 1, \dots, n)$.*

Proposition 2.13. *The set $\Omega = \{(z, t) \in C^{2n} \times R \mid |z|^{4\sigma} + t^2 < 1\}$ is a characteristic domain with respect to vector fields $X_j, Y_j (j = 1, \dots, n)$.*

Proof. Let $\varphi(z, t) = |z|^{4\sigma} + t^2 - 1$, then

$$\begin{aligned} X_j \varphi(z, t) &= \frac{\partial}{\partial x_j} (|z|^{4\sigma} + t^2 - 1) + 2\sigma y_j |z|^{2\sigma-2} \frac{\partial}{\partial t} (|z|^{4\sigma} + t^2 - 1) \\ &= 4\sigma |z|^{4\sigma-2} x_j + 4\sigma y_j |z|^{2\sigma-2} t; \\ Y_j \varphi(z, t) &= \frac{\partial}{\partial y_j} (|z|^{4\sigma} + t^2 - 1) - 2\sigma x_j |z|^{2\sigma-2} \frac{\partial}{\partial t} (|z|^{4\sigma} + t^2 - 1) \\ &= 4\sigma |z|^{4\sigma-2} y_j - 4\sigma x_j |z|^{2\sigma-2} t. \end{aligned}$$

Hence

$$\begin{aligned} |\nabla_L \varphi(z, t)|^2 &= \sum_{j=1}^n (|X_j \varphi(z, t)|^2 + |Y_j \varphi(z, t)|^2) \\ &= \sum_{j=1}^n (16\sigma^2 |z|^{8\sigma-4} (x_j^2 + y_j^2) + 16\sigma^2 |z|^{4\sigma-4} t^2 (x_j^2 + y_j^2)) \\ &= 16\sigma^2 |z|^{8\sigma-2} + 16\sigma^2 |z|^{4\sigma-2} t^2. \end{aligned}$$

If $|\nabla_L \varphi(z, t)| = 0$, then $|z| = 0$. We see that two points satisfying $z = 0, t = \pm 1$ are on the boundary $\partial\Omega$ and they are characteristic points. \square

Corollary 2.14. *The sets $\Omega_r = \{(z, t) \in C^{2n} \times R \mid |z|^{4\sigma} + t^2 < r^{4\sigma}\} (r > 0)$ are characteristic domains with characteristic points $(0, \pm r^{2\sigma})$.*

3. THE PROOFS OF THEOREM 1.1 AND COROLLARIES 1.2-1.4

Proof of Theorem 1.1. We apply (2.3) with $A = L = -\Delta_L$, $B_1 = x_1, \dots, B_n = x_n, B_{n+1} = y_1, \dots, B_{2n} = y_n, T_1 = X_1, \dots, T_n = X_n, T_{n+1} = Y_1, \dots, T_{2n} = Y_n$, $f(x) = (\lambda - x)^\alpha$, $g(x) = (\lambda - x)^\beta$, and obtain

$$\begin{aligned} (3.1) \quad & \left(\sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^\alpha (\langle [X_p, x_p] u_i, u_i \rangle_{L^2} + \langle [Y_p, y_p] u_i, u_i \rangle_{L^2}) \right)^2 \\ & \leq 4 \left(\sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^\beta (\langle [L, x_p] u_i, x_p u_i \rangle_{L^2} + \langle [L, y_p] u_i, y_p u_i \rangle_{L^2}) \right) \\ & \quad \times \left(\sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} (\|X_p u_i\|_{L^2}^2 + \|Y_p u_i\|_{L^2}^2) \right). \end{aligned}$$

Since

$$[X_p, x_p] u_i = [Y_p, y_p] u_i = u_i,$$

and

$$\langle [L, x_p] u_i, x_p u_i \rangle_{L^2} = 2 \int_{\Omega} u_i^2 - \langle [L, x_p] u_i, x_p u_i \rangle_{L^2}$$

from (2.2), it follows

$$(3.2) \quad \langle [L, x_p] u_i, x_p u_i \rangle_{L^2} = \int_{\Omega} u_i^2 = 1.$$

In a similar way, we obtain

$$(3.3) \quad \langle [L, y_p] u_i, y_p u_i \rangle_{L^2} = \int_{\Omega} u_i^2 = 1.$$

On the other hand, it yields

$$(3.4) \quad \sum_{p=1}^n \|X_p u_i\|_{L^2}^2 + \sum_{p=1}^n \|Y_p u_i\|_{L^2}^2 = \int_{\Omega} \nabla_L u_i \nabla_L u_i = \int_{\Omega} L u_i u_i = \int_{\Omega} \lambda_i u_i u_i = \lambda_i.$$

Instituting (3.2), (3.3) and (3.4) into (3.1), it deduces (1.4). \square

Proof of Corollary 1.2. To obtain (1.5), we only need to take $\alpha = \beta = 2$ in (1.4). \square

Proof of Corollary 1.3. When $\alpha = \beta$, we have from Theorem 1.1 that

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha} \leq \frac{2}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \lambda_i.$$

Using Lemmas 2.6 and 2.7, it implies

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha} \geq \frac{1}{k^{\alpha-1}} \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \right)^{\alpha} \geq \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \right)^{\alpha-1} (\lambda_{k+1} - \lambda_k)$$

and

$$\frac{2}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \lambda_i \leq \frac{2}{nk} \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \right) \left(\sum_{i=1}^k \lambda_i \right),$$

hence

$$\left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \right)^{\alpha-1} (\lambda_{k+1} - \lambda_k) \leq \frac{2}{nk} \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \right) \left(\sum_{i=1}^k \lambda_i \right).$$

Since

$$\left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i) \right)^{\alpha-1} \geq \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1},$$

it shows (1.6). \square

Proof of Corollary 1.4. When $1 \leq \alpha = \beta \leq 2$, we have from Theorem 1.1 that

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha} \leq \frac{2}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \lambda_i,$$

then

$$\begin{aligned}
& \lambda_{k+1} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} - \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \lambda_i \\
&= \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} (\lambda_{k+1} - \lambda_i) \\
&\leq \frac{2}{n} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \lambda_i,
\end{aligned}$$

or

$$\begin{aligned}
& \lambda_{k+1} \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \\
&\leq \left(1 + \frac{2}{n}\right) \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \lambda_i \\
&\leq \left(1 + \frac{2}{n}\right) \frac{1}{k} \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \right) \left(\sum_{i=1}^k \lambda_i \right),
\end{aligned}$$

where Lemma 2.7 is used. Therefore

$$\left(\lambda_{k+1} - \left(1 + \frac{2}{n}\right) \frac{1}{k} \left(\sum_{i=1}^k \lambda_i \right) \right) \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \right) \leq 0.$$

Since $\left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \right) \geq 0$, it follows

$$\lambda_{k+1} - \left(1 + \frac{2}{n}\right) \frac{1}{k} \left(\sum_{i=1}^k \lambda_i \right) \leq 0$$

and (1.7) is proved. \square

4. PROOFS OF THEOREM 1.5 AND COROLLARIES 1.6-1.9

Proof of Theorem 1.5. Applying (2.3) with $A = L^2 = (-\Delta_L)^2$, $B_1 = x_1, \dots, B_n = x_n, B_{n+1} = y_1, \dots, B_{2n} = y_n, T_1 = X_1, \dots, T_n = X_n, T_{n+1} = Y_1, \dots, T_{2n} = Y_n$, $f(x) = (\lambda - x)^\alpha$, $g(x) = (\lambda - x)^\beta$, it follows

$$\begin{aligned}
(4.1) \quad & \left(\sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^\alpha (\langle [X_p, x_p] u_i, u_i \rangle_{L^2} + \langle [Y_p, y_p] u_i, u_i \rangle_{L^2}) \right)^2 \\
& \leq 4 \left(\sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^\beta (\langle [L^2, x_p] u_i, x_p u_i \rangle_{L^2} + \langle [L^2, y_p] u_i, y_p u_i \rangle_{L^2}) \right) \\
& \quad \times \left(\sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} (\|X_p u_i\|_{L^2}^2 + \|Y_p u_i\|_{L^2}^2) \right).
\end{aligned}$$

Since

$$\begin{aligned} & \sum_{p=1}^n \|X_p u_i\|_{L^2}^2 + \sum_{p=1}^n \|Y_p u_i\|_{L^2}^2 \\ &= \int_{\Omega} \nabla_L u_i \nabla_L u_i = \int_{\Omega} L u_i \cdot u_i \\ &\leq \left(\int_{\Omega} u_i^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} (L u_i)^2 \right)^{\frac{1}{2}} = \lambda_i^{\frac{1}{2}}, \end{aligned}$$

it implies

$$\begin{aligned} (4.2) \quad & \left(\sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \left(\|X_p u_i\|_{L^2}^2 + \|Y_p u_i\|_{L^2}^2 \right) \right) \\ &= \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \lambda_i^{\frac{1}{2}} \right). \end{aligned}$$

Recalling (3.2) and (3.3), we have

$$\begin{aligned} (4.3) \quad & \left(\sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^{\alpha} \left(\langle [X_p, x_p] u_i, u_i \rangle_{L^2} + \langle [Y_p, y_p] u_i, u_i \rangle_{L^2} \right) \right)^2 \\ &= 4n^2 \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha} \right)^2. \end{aligned}$$

On the other hand, it obtains by (2.2) that

$$\begin{aligned} [L^2, x_p] u_i &= L^2 (x_p u_i) - x_p L^2 u_i \\ &= -2X_p L u_i - 2L (X_p u_i), \end{aligned}$$

and

$$[L^2, y_p] u_i = -2Y_p L u_i - 2L (Y_p u_i).$$

Hence, we have

$$\begin{aligned} \langle [L^2, x_p] u_i, x_p u_i \rangle_{L^2} &= 2 \int_{\Omega} L u_i \cdot X_p (x_p u_i) - 2 \int_{\Omega} x_p X_p u_i \cdot L u_i - 4 \int_{\Omega} X_p^2 u_i \cdot u_i \\ &= 2 \int_{\Omega} L u_i \cdot u_i - 4 \int_{\Omega} X_p^2 u_i \cdot u_i \end{aligned}$$

and

$$\begin{aligned} \langle [L^2, y_p] u_i, y_p u_i \rangle_{L^2} &= 2 \int_{\Omega} L u_i \cdot Y_p (y_p u_i) - 2 \int_{\Omega} y_p Y_p u_i \cdot L u_i - 4 \int_{\Omega} Y_p^2 u_i \cdot u_i \\ &= 2 \int_{\Omega} L u_i \cdot u_i - 4 \int_{\Omega} Y_p^2 u_i \cdot u_i. \end{aligned}$$

Noting

$$- \sum_{p=1}^n \int_{\Omega} X_p^2 u_i \cdot u_i - \sum_{p=1}^n \int_{\Omega} Y_p^2 u_i \cdot u_i = \sum_{p=1}^n \|X_p u_i\|_{L^2}^2 + \sum_{p=1}^n \|Y_p u_i\|_{L^2}^2 = \int_{\Omega} L u_i \cdot u_i,$$

so

$$\begin{aligned}
(4.4) \quad & \sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^\beta \left(\langle [L^2, x_p] u_i, x_p u_i \rangle_{L^2} + \langle [L^2, y_p] u_i, y_p u_i \rangle_{L^2} \right) \\
&= \sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^\beta \left(2 \int_{\Omega} L u_i \cdot u_i - 4 \int_{\Omega} X_p^2 u_i \cdot u_i \right) \\
&\quad + \sum_{i=1}^k \sum_{p=1}^n (\lambda_{k+1} - \lambda_i)^\beta \left(2 \int_{\Omega} L u_i \cdot u_i - 4 \int_{\Omega} Y_p^2 u_i \cdot u_i \right) \\
&= 4(n+1) \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\beta \int_{\Omega} L u_i \cdot u_i \\
&\leq 4(n+1) \sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\beta \lambda_i^{\frac{1}{2}}.
\end{aligned}$$

Taking (4.2), (4.3) and (4.4) into (4.1), we obtain (1.8). \square

Proof of Corollary 1.6. To obtain (1.9), take $\alpha = \beta = 2$ in (1.8). \square

Proof of Corollary 1.7. From Theorem 1.5, we have

$$\left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \right)^2 \leq \frac{4(n+1)}{n^2} \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\beta \lambda_i^{\frac{1}{2}} \right) \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \lambda_i^{\frac{1}{2}} \right).$$

Applying Lemma 2.8 with $A_i = \lambda_{k+1} - \lambda_i$ and $B_i = C_i = \lambda_i^{\frac{1}{2}}$, it deduces (1.10). \square

Proof of Corollary 1.8. To obtain (1.11), we only need to take $\alpha = \beta = 2$ in Corollary 1.7. \square

Proof of Corollary 1.9. We have from (1.8) that

$$\left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \right)^2 \leq \frac{4(n+1)}{n^2} \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\beta \lambda_i^{\frac{1}{2}} \right) \times \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \lambda_i^{\frac{1}{2}} \right).$$

Applying Lemma 2.7 to $\left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\beta \lambda_i^{\frac{1}{2}} \right)$ and $\left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \lambda_i^{\frac{1}{2}} \right)$, it follows

$$\begin{aligned}
& \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \right)^2 \\
& \leq \frac{4(n+1)}{n^2 k^2} \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\beta \right) \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{2\alpha-\beta-1} \right) \left(\sum_{i=1}^k \lambda_i^{\frac{1}{2}} \right)^2 \\
& = \frac{4(n+1)}{n^2 k^2} \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \right) \times \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \right) \left(\sum_{i=1}^k \lambda_i^{\frac{1}{2}} \right)^2,
\end{aligned}$$

where we have used $1 \leq \alpha = \beta \leq 2$. It implies

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^\alpha \leq \frac{4(n+1)}{n^2 k^2} \left(\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \right) \left(\sum_{i=1}^k \lambda_i^{\frac{1}{2}} \right)^2,$$

then

$$\sum_{i=1}^k (\lambda_{k+1} - \lambda_i)^{\alpha-1} \left((\lambda_{k+1} - \lambda_k) - \frac{4(n+1)}{n^2 k^2} \left(\sum_{i=1}^k \lambda_i^{\frac{1}{2}} \right)^2 \right) \leq 0,$$

since $\lambda_i \leq \lambda_k$ for all $i \leq k$. Hence

$$(\lambda_{k+1} - \lambda_k) - \frac{4(n+1)}{n^2 k^2} \left(\sum_{i=1}^k \lambda_i^{\frac{1}{2}} \right)^2 \leq 0,$$

and (1.12) is proved. \square

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